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A note on the Picard motive of a variety

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Abstract

We prove that the Picard motive of a smooth projective variety and the Picard motive of its Albanese variety are isomorphic, under the assumption that both the variety and its Albanese variety have dimension at least 2.

Keywords: Chow motives; Picard motive; Albanese variety

1. Introduction

For various Weil cohomology theories ($H^*(\cdot)$, e.g., singular cohomology, étale cohomology, etc.), we have the well-known isomorphism

$$H^1(X) \simeq H^1(J_X),$$

where X is a smooth projective variety and J_X is its Albanese variety, both defined over some field k. Therefore, it is natural to ask if a similar isomorphism holds in the category of pure Chow motives ($\mathcal{M}$). More precisely, if we let $h^1(X)$ and $h^1(J_X)$ denote the Picard motives of X and J_X, respectively [1,2], is there an isomorphism

$$h^1(X) \stackrel{?}{\simeq} h^1(J_X) \tag{1}$$

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In this note, we prove that, as expected, we have such an isomorphism, under the assumption that both the variety X and its Albanese variety J_X have dimension at least 2. In fact, this isomorphism is provided by the isomorphism of the Picard varieties of X and J_X.

2. Notation and preliminaries

The Picard motive was first introduced by Murre in [1]. Later, Scholl gave an excellent exposition of the theory in [2]. Here, we will follow Scholl's notation with minor exceptions. We write P_X for the Picard variety of X, and J_X (or A) for its Albanese variety (i.e., the dual of P_X). We denote by d_X the dimension of X and by d_A the dimension J_X. They are both assumed to be greater than 1. For a morphism of varieties $\phi : X \to Y$, we denote by J_ϕ (respectively, P_ϕ) the induced morphism $J_X \to J_Y$ (respectively, $P_Y \to P_X$); $[\Gamma_\phi]$ denotes the (rational) equivalence class of the graph of ϕ, and $[\Gamma_\phi]^t$ denotes the class of the transpose of its graph. If we let $\sigma : X \to J_X$ denote the Albanese map (after fixing a point $x_0 \in X$ and $\sigma(x_0) = e$, the identity element with respect to the group operation on J_X), we have $J_\sigma = \mathrm{id}_{J_X}$, and $P_\sigma : P_{J_X} \xrightarrow{\sim} P_X$ is an isomorphism.

For the sake of completeness, we include some results of Scholl here.

Theorem 1 (*[2, Theorem 3.9]*). *Let X and Y be smooth, projective varieties (defined over some field k) of dimensions d_X and d_Y, respectively.*

1. *Then there is an isomorphism*

$$\mathrm{Hom}_{Ab}(J_X, P_Y) \otimes \mathbb{Q} \xrightarrow{\sim} \frac{A^1(X \times Y)}{p_1^* A^1(X) + p_2^* A^1(Y)},$$

where $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ are projections and $A^1(\cdot)$ denotes the Chow group of codimension 1 cycles with rational coefficients.

2. *Let $\zeta \in A^{d_X}(X)$ and $\eta \in A^{d_Y}(Y)$ be fixed zero cycles of positive degree. Then there is an isomorphism*

$$\mathrm{Hom}_{Ab}(J_X, P_Y) \otimes \mathbb{Q} \xrightarrow{\Omega} \{c \in A^1(X \times Y) | c \circ \zeta_* = 0 = \eta^* \circ c\}.$$

(For definitions of ζ_* and η^*, see [2, p.169].)

Proposition 1 (*[2, Proposition 3.10]*). *Let X, Y, X' and Y' be smooth, projective varieties and $\phi : X' \to X$, $\psi : Y' \to Y$ and $\mu : J_X \to P_Y$ be morphisms. Then*

$$\Omega(P_\psi \circ \mu) = [\Gamma_\psi]^t \circ \Omega(\mu) \quad \text{and} \quad \Omega(\mu \circ J_\phi) = \Omega(\mu) \circ [\Gamma_\phi],$$

where Ω is the isomorphism in the Theorem 1.

Let C_X and C_A be one-dimensional linear sections of X and J_X, respectively. Then

$$\xi_X := [\Gamma_{i_X}] \circ [\Gamma_{i_X}]^t \quad \text{and} \quad \xi_A := [\Gamma_{i_A}] \circ [\Gamma_{i_A}]^t,$$

where $i_X : C_X \hookrightarrow X$ and $i_A : C_A \hookrightarrow J_X$ denote the embedding maps. We have the induced isogenies

$$\alpha_X : P_X \xrightarrow{P_{i_X}} P_{C_X} = J_{C_X} \xrightarrow{J_{i_X}} J_X$$

$$\alpha_A : P_{J_X} \xrightarrow{P_{i_A}} P_{C_A} = J_{C_A} \xrightarrow{J_{i_A}} J_{J_X} = J_X.$$

We will denote the inverse isogenies by β_X and β_A, respectively. By taking a suitable common multiple if necessary, we can write

$$\alpha_X \circ \beta_X = n \cdot id_{J_X} = \alpha_A \circ \beta_A.$$

Then, the correspondences

$$p_1^X := \frac{1}{n}\Omega(\beta_X) \circ \xi_X, \qquad p_{2d_X-1}^X := (p_1^X)^t, \qquad \pi_1^X := p_1^X \circ \left(1 - \frac{1}{2}p_{2d_X-1}^X\right)$$

are projectors, and the Picard motive is given by

$$h^1(X) := (X, \pi_1^X).$$

In fact, $\pi_1^X = p_1^X$ when $d_X \geq 3$ [2].

3. The isomorphism

We would like to prove that $h^1(X)$ and $h^1(J_X)$ are isomorphic in $\mathcal{M}$. In fact, it is enough to prove the isomorphism of (X, p_1^X) and (J_X, p_1^A), as $\pi_1^X \circ p_1^X : (X, p_1^X) \to h^1(X)$ and $p_1^X \circ \pi_1^X : h^1(X) \to (X, p_1^X)$ are mutually inverse isomorphisms (of course, the same holds for p_1^A and π_1^A). We define

$$\theta_1 := \frac{1}{n} p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \in \mathrm{Hom}_{\mathcal{M}}((X, p_1^X), (J_X, p_1^A))$$
$$\theta_2 := \frac{1}{n} p_1^X \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \in \mathrm{Hom}_{\mathcal{M}}((J_X, p_1^A), (X, p_1^X)).$$

Note that, in the definition of θ_2, we interpret β_X as a morphism $J_{J_X} \to P_X$, namely $\beta_X = \beta_X \circ J_\sigma^{-1}$ but $J_\sigma = id_{J_X}$.

We will show that θ_1 and θ_2 are mutually inverse isomorphisms.

$$\begin{aligned}
\theta_2 \circ \theta_1 &= \frac{1}{n^2} p_1^X \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \\
&= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X) \circ [\Gamma_{i_A}] \circ [\Gamma_{i_A}]^t \circ \Omega(\beta_A) \circ [\Gamma_{i_A}] \circ [\Gamma_{i_A}]^t \\
&\quad \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \\
&= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X \circ J_{i_A}) \circ \Omega(P_{i_A} \circ \beta_A \circ J_{i_A}) \circ \Omega(P_{i_A} \circ \beta_A \circ J_\sigma) \circ \xi_X \circ p_1^X \\
&= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X \circ J_{i_A} \circ P_{i_A} \circ \beta_A \circ J_{i_A} \circ P_{i_A} \circ \beta_A \circ id_{J_X}) \circ \xi_X \circ p_1^X \\
&= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X \circ \alpha_A \circ \beta_A \circ \alpha_A \circ \beta_A) \circ \xi_X \circ p_1^X \\
&= p_1^X \circ p_1^X \circ p_1^X = p_1^X.
\end{aligned}$$

For the converse, we have

$$\begin{aligned}
\theta_1 \circ \theta_2 &= \frac{1}{n^2} p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \\
&= \frac{1}{n^3} p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ [\Gamma_{i_X}] \circ [\Gamma_{i_X}]^t
\end{aligned}$$

$$
\begin{aligned}
&\circ \Omega(\beta_X) \circ [\Gamma_{i_X}] \circ [\Gamma_{i_X}]^t \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \\
&= \frac{1}{n^3} p_1^A \circ \Omega(\beta_A \circ J_\sigma \circ J_{i_X}) \circ \Omega(P_{i_X} \circ \beta_X \circ J_{i_X}) \circ \Omega(P_{i_X} \circ \beta_X) \circ \xi_A \circ p_1^A \\
&= \frac{1}{n^3} p_1^A \circ \Omega(\beta_A \circ id_{J_X} \circ \alpha_X \circ \beta_X \circ \alpha_X \circ \beta_X) \circ \xi_A \circ p_1^A \\
&= p_1^A \circ p_1^A \circ p_1^A = p_1^A.
\end{aligned}
$$

Hence, the isomorphism (1) holds.

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